CIVE2400 Fluid Mechanics

Section 1: Fluid Flow in Pipes

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1. Fluid Flow in Pipes

We will be looking here at the flow of real fluid in pipes – *real* meaning a fluid that looses energy due to friction as it interacts with the pipe wall as it flows.

Recall from Level 1 that the shear stress induced in a fluid flowing near a boundary is given by Newton's law of viscosity:

$$\tau \propto \frac{du}{dy}$$

This tells us that the shear stress, τ, in a fluid is proportional to the velocity gradient - the rate of change of velocity across the fluid path. For a "Newtonian" fluid we can write:

$$\tau = \mu \frac{du}{dy}$$

where the constant of proportionality, μ, is known as the coefficient of viscosity (or simply viscosity).

Recall also that flow can be classified into one of two types, **laminar** or **turbulent** flow (with a small transitional region between these two). The non-dimensional number, the Reynolds number, Re, is used to determine which type of flow occurs:

$$\mathrm{Re} = \frac{\rho u d}{\mu}$$

Laminar flow:	$\mathrm{Re} < 2000$
Transitional flow:	$2000 < \mathrm{Re} < 4000$
Turbulent flow:	$\mathrm{Re} > 4000$

It is important to determine the flow type as this governs how the amount of energy lost to friction relates to the speed of the flow. And hence how much energy must be used to move the fluid.

1.1 Pressure loss due to friction in a pipeline.

Consider a cylindrical element of incompressible fluid flowing in the pipe, as shown

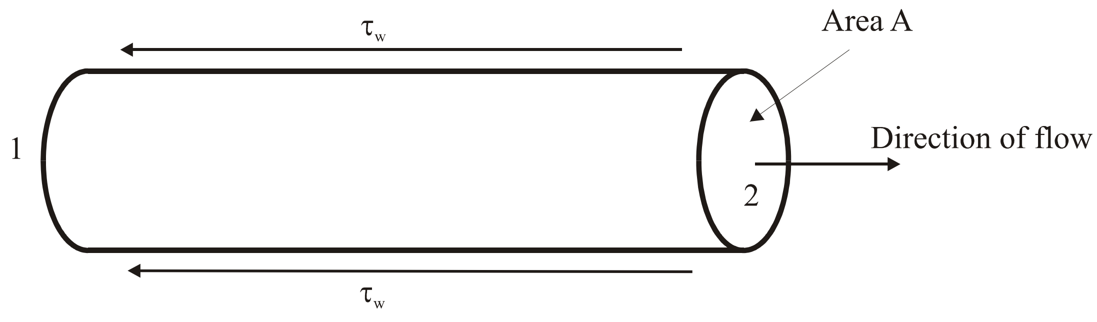

Figure 1: Element of fluid in a pipe

The pressure at the upstream end, 1, is p, and at the downstream end, 2, the pressure has fallen by Δp to (p-Δp).

The driving force due to pressure (F = Pressure x Area) can then be written

driving force = Pressure force at 1 - pressure force at 2

$$pA - (p - \Delta p)A = \Delta p\, A = \Delta p \frac{\pi d^2}{4}$$

The retarding force is that due to the shear stress by the walls

= shear stress $\times$ area over which it acts

= $\tau_w \times$ area of pipe wall

= $\tau_w \pi d L$

As the flow is in equilibrium,

driving force = retarding force

$$\Delta p \frac{\pi d^2}{4} = \tau_w \pi d L$$

$$\Delta p = \frac{\tau_w 4L}{d}$$

Equation 1

Giving an expression for pressure loss in a pipe in terms of the pipe diameter and the shear stress at the wall on the pipe.

The shear stress will vary with velocity of flow and hence with Re. Many experiments have been done with various fluids measuring the pressure loss at various Reynolds numbers. These results plotted to show a graph of the relationship between pressure loss and Re look similar to the figure below:

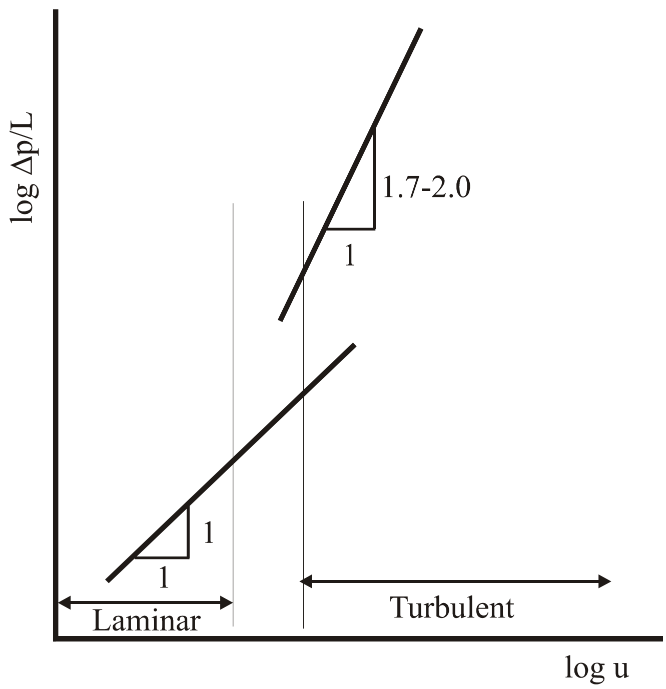

Figure 2: Relationship between velocity and pressure loss in pipes

This graph shows that the relationship between pressure loss and Re can be expressed as

laminar $\Delta p \propto u$

turbulent $\Delta p \propto u^{1.7 \text{ (or } 2.0)}$

As these are empirical relationships, they help in determining the pressure loss but not in finding the magnitude of the shear stress at the wall τ_w on a particular fluid. If we knew τ_w we could then use it to give a general equation to predict the pressure loss.

1.2 Pressure loss during laminar flow in a pipe

In general the shear stress τ_w. is almost impossible to measure. But for laminar flow it is possible to calculate a theoretical value for a given velocity, fluid and pipe dimension. (As this was covered in he Level 1 module, only the result is presented here.) The pressure loss in a pipe with laminar flow is given by the Hagen-Poiseuille equation:

$$\Delta p = \frac{32\mu L u}{d^2}$$

or in terms of head

$$h_f = \frac{32\mu L u}{\rho g d^2}$$

Equation 2

Where $\boldsymbol{h_f}$ is known as the ***head-loss due to friction***

1.3 Pressure loss during turbulent flow in a pipe

In this derivation we will consider a general bounded flow - fluid flowing in a channel - we will then apply this to pipe flow. In general it is most common in engineering to have Re > 2000 i.e. turbulent flow – in both closed (pipes and ducts) and open (rivers and channels). However analytical expressions are not available so empirical relationships are required (those derived from experimental measurements).

Consider the element of fluid, shown in figure 3 below, flowing in a channel, it has length L and with wetted perimeter P. The flow is steady and uniform so that acceleration is zero and the flow area at sections 1 and 2 is equal to A.

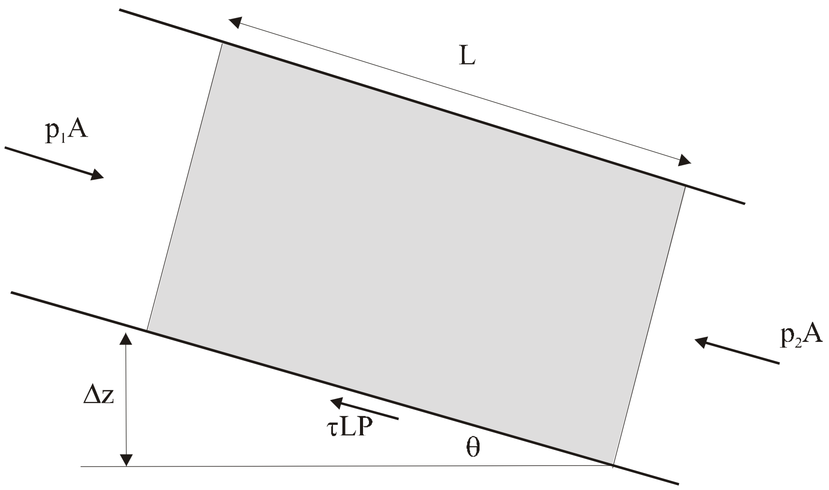

Figure 3: Element of fluid in a channel flowing with uniform flow

$$p_1 A - p_2 A - \tau_w LP + W\sin\theta = 0$$

writing the weight term as ρgAL and sin θ = –Δz/L gives

$$A(p_1 - p_2) - \tau_w LP - \rho gA\Delta z = 0$$

this can be rearranged to give

$$\frac{[(p_1 - p_2) - \rho g\Delta z]}{L} - \tau_o \frac{P}{A} = 0$$

where the first term represents the piezometric head loss of the length L or (writing piezometric head p^*)

$$\tau_o = m\frac{dp^*}{dx}$$

Equation 3

where $m = A/P$ is known as the hydraulic mean depth

Writing piezometric head loss as $p^* = \rho gh_f$, then shear stress per unit length is expressed as

$$\tau_o = m\frac{dp^*}{dx} = m\frac{\rho gh_f}{L}$$

So we now have a relationship of shear stress at the wall to the rate of change in piezometric pressure. To make use of this equation an empirical factor must be introduced. This is usually in the form of a **friction factor *f***, and written

$$\tau_o = f\frac{\rho u^2}{2}$$

where u is the mean flow velocity.

Hence

$$\frac{dp^*}{dx} = f\frac{\rho u^2}{2m} = \frac{\rho gh_f}{L}$$

So, for a general bounded flow, head loss due to friction can be written

$$h_f = \frac{fLu^2}{2m}$$

Equation 4

More specifically, for a circular pipe, $m = A/P = \pi d^2/4\pi d = d/4$ giving

$$h_f = \frac{4fLu^2}{2gd}$$

Equation 5

This is known as the **Darcy-Weisbach** equation for head loss in circular pipes

(Often referred to as the Darcy equation)

This equation is equivalent to the Hagen-Poiseuille equation for laminar flow with the exception of the empirical friction factor f introduced.

It is sometimes useful to write the Daryc equation in terms of discharge Q, (using $Q = Au$)

$$u = \frac{4Q}{\pi d^2}$$

$$h_f = \frac{64 fLQ^2}{2g\pi^2 d^5} = \frac{fLQ^2}{3.03d^5}$$

Equation 6

Or with a 1% error

$$h_f = \frac{fLQ^2}{3d^5}$$

Equation 7

NOTE On Friction Factor Value

The f value shown above is different to that used in American practice. Their relationship is

$$f_{American} = 4f$$

Sometimes the f is replaced by the Greek letter λ. where

$$\lambda = f_{American} = 4f$$

Consequently great care must be taken when choosing the value of f with attention taken to the source of that value.

1.4 Choice of friction factor f

The value of f must be chosen with care or else the head loss will not be correct. Assessment of the physics governing the value of friction in a fluid has led to the following relationships

1. $h_f \propto L$
2. $h_f \propto v^2$
3. $h_f \propto 1/d$
4. h_f depends on surface roughness of pipes
5. h_f depends on fluid density and viscosity
6. h_f is independent of pressure

Consequently f cannot be a constant if it is to give correct head loss values from the Darcy equation. An expression that gives f based on fluid properties and the flow conditions is required.

1.4.1 The value of f for Laminar flow

As mentioned above the equation derived for head loss in turbulent flow is equivalent to that derived for laminar flow – the only difference being the empirical *f*. Equation the two equations for head loss allows us to derive an expression of *f* that allows the Darcy equation to be applied to laminar flow.

Equating the Hagen-Poiseuille and Darcy-Weisbach equations gives:

$$\frac{32\mu Lu}{\rho g d^2} = \frac{4fLu^2}{2gd}$$

$$f = \frac{16\mu}{\rho v d}$$

$$f = \frac{16}{\text{Re}}$$

Equation 8

1.4.2 Blasius equation for f

Blasius, in 1913, was the first to give an accurate empirical expression for *f* for turbulent flow in smooth pipes, that is:

$$f = \frac{0.079}{\text{Re}^{0.25}}$$

Equation 9

This expression is fairly accurate, giving head losses +/- 5% of actual values for Re up to 100000.

1.4.3 Nikuradse

Nikuradse made a great contribution to the theory of pipe flow by differentiating between rough and smooth pipes. A rough pipe is one where the mean height of roughness is greater than the thickness of the laminar sub-layer. Nikuradse artificially roughened pipe by coating them with sand. He defined a *relative roughness* value k_s/d (mean height of roughness over pipe diameter) and produced graphs of *f* against Re for a range of relative roughness 1/30 to 1/1014.

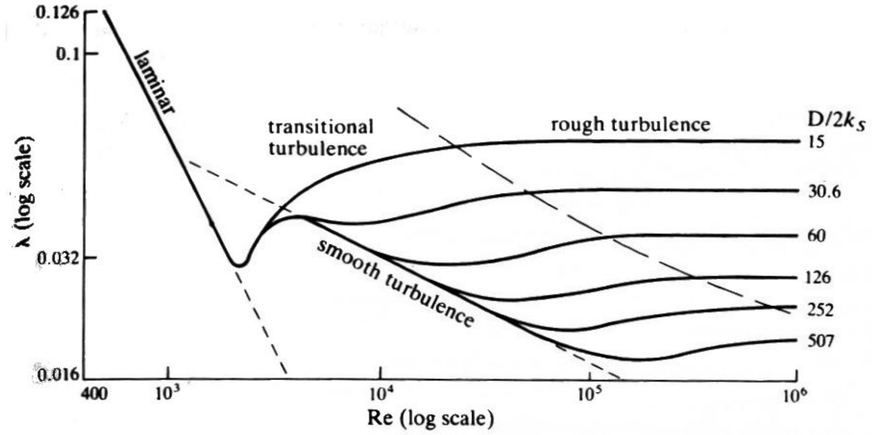

Figure 4: Regions on plot of Nikurades's data

A number of distinct regions can be identified on the diagram.

The regions which can be identified are:

1. Laminar flow (f = 16/Re)
2. Transition from laminar to turbulent
   A unstable region between Re = 2000 and 4000. Pipe flow normally lies outside this region
3. Smooth turbulent
   The limiting line of turbulent flow. All value of relative roughness tend toward this as Re decreases.
4. Transitional turbulent
   The region which f varies with both Re and relative roughness. Most pipes lie in this region.
5. Rough turbulent. f remains constant for a given relative roughness. It is independent of Re.

1.4.4 Colebrook-White equation for f

Colebrook and White did a large number of experiments on commercial pipes and they also brought together some important theoretical work by von Karman and Prandtl. This work resulted in an equation attributed to them as the Colebrook-White equation:

$$\frac{1}{\sqrt{f}} = -4\log_{10}\left(\frac{k_s}{3.71d} + \frac{1.26}{\mathrm{Re}\sqrt{f}}\right)$$

Equation 10

It is applicable to the whole of the turbulent region for commercial pipes and uses an effective roughness value (k_s) obtained experimentally for all commercial pipes.

Note a particular difficulty with this equation. f appears on both sides in a square root term and so cannot be calculated easily. Trial and error methods must be used to get f once k_s,Re and d are known. (In the 1940s when calculations were done by slide rule this was a time consuming task.) Nowadays it is relatively trivial to solve the equation on a programmable calculator of spreadsheet.

Moody made a useful contribution to help, he potted f against Re for commercial pipes – see the figure below. This figure has become known as the Moody Diagram

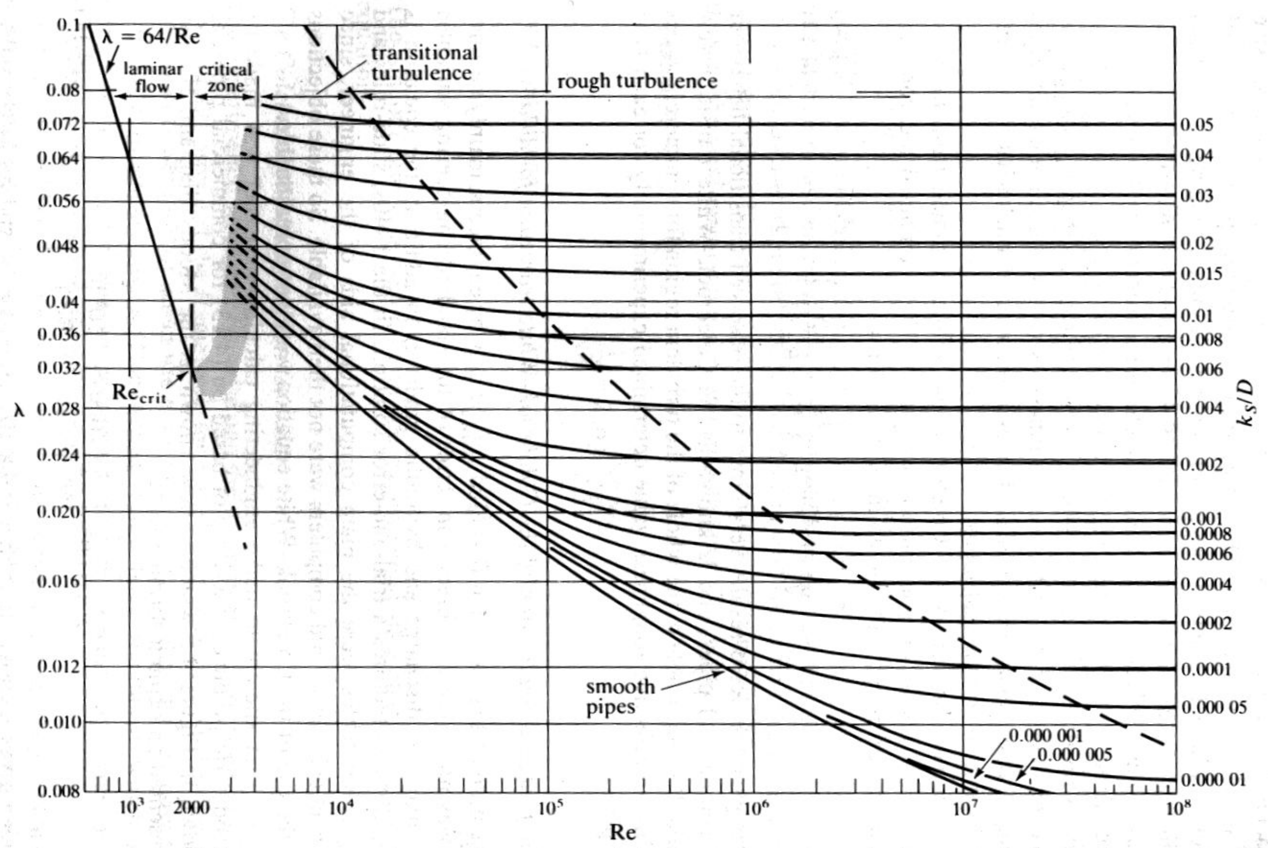

Figure 5: Moody Diagram.

He also produced an equation based on the Colebrook-White equation that made it simpler to calculate f:

$$f = 0.001375\left[1+\left(\frac{200k_s}{d}+\frac{10^6}{\mathrm{Re}}\right)^{1/3}\right]$$

Equation 11

This equation of Moody gives f correct to +/- 5% for $4 \times 10^3 < \mathrm{Re} < 1 \times 10^7$ and for $k_s/d < 0.01$.

Barr presented an alternative explicit equation for f in 1975

$$\frac{1}{\sqrt{f}} = -4\log_{10}\left[\frac{k_s}{3.71d}+\frac{5.1286}{\mathrm{Re}^{0.89}}\right]$$

Equation 12

or

$$f = 1\Big/\left[-4\log_{10}\left(\frac{k_s}{3.71d}+\frac{5.1286}{\mathrm{Re}^{0.89}}\right)\right]^2$$

Equation 13

Here the last term of the Colebrook-White equation has been replaced with $5.1286/\mathrm{Re}^{0.89}$ which provides more accurate results for $\mathrm{Re} > 10^5$.

The problem with these formulas still remains that these contain a dependence on k_s. What value of k_s should be used for any particular pipe? Fortunately pipe manufactures provide values and typical values can often be taken similar to those in table 1 below.

Pipe Material	k_s (mm)
Brass, copper, glass, Perspex	0.003
Asbestos cement	0.03
Wrought iron	0.06
Galvanised iron	0.15
Plastic	0.03
Bitumen-lined ductile iron	0.03
Spun concrete lined ductile iron	0.03
Slimed concrete sewer	6.0

Table 1: Typical k_s values

1.5 Local Head Losses

In addition to head loss due to friction there are always head losses in pipe lines due to bends, junctions, valves etc. (See notes from Level 1, Section 4 - Real Fluids for a discussion of energy losses in flowing fluids.) For completeness of analysis these should be taken into account. In practice, in long pipe lines of several kilometres their effect may be negligible for short pipeline the losses may be greater than those for friction.

A general theory for local losses is not possible, however rough turbulent flow is usually assumed which gives the simple formula

$$h_L = k_L \frac{u^2}{2g}$$

Equation 14

Where h_L is the local head loss and k_L is a constant for a particular fitting (valve or junction etc.)

For the cases of sudden contraction (e.g. flowing out of a tank into a pipe) of a sudden enlargement (e.g. flowing from a pipe into a tank) then a theoretical value of k_L can be derived. For junctions bend etc. k_L must be obtained experimentally.

1.5.1 Losses at Sudden Enlargement

Considerthe flow in the sudden enlargement, shown in figure 6 below, fluid flows from section 1 to section 2. The velocity must reduce and so the pressure increases (as follows from Bernoulli). At position 1' turbulent eddies occur which give rise to the local head loss.

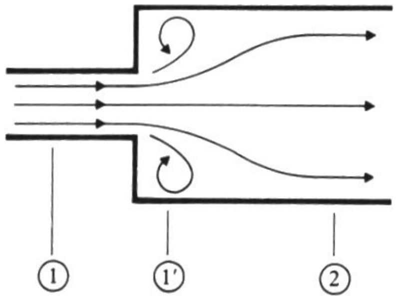

Figure 6: Sudden Expansion

Apply the momentum equation between positions 1 and 2 to give:

$$p_1 A_1 - p_2 A_2 = \rho Q(u_2 - u_1)$$

Now use the continuity equation to remove Q. (i.e. substitute $Q = A_2 u_2$)

$$p_1 A_1 - p_2 A_2 = \rho A_2 u_2 (u_2 - u_1)$$

Rearranging gives

$$\frac{p_2 - p_1}{\rho g} = \frac{u_2}{g}(u_1 - u_2)$$

Equation 17

Now apply the Bernoulli equation from point 1 to 2, with the head loss term h_L

$$\frac{p_1}{\rho g} + \frac{u_1^2}{2g} = \frac{p_2}{\rho g} + \frac{u_2^2}{2g} + h_L$$

And rearranging gives

$$h_L = \frac{u_1^2 - u_2^2}{2g} - \frac{p_2 - p_1}{\rho g}$$

Equation 18

Combining Equations 17 and 18 gives

$$h_L = \frac{u_1^2 - u_2^2}{2g} - \frac{u_2}{g}(u_1 - u_2)$$

$$h_L = \frac{(u_1 - u_2)^2}{2g}$$

Equation 19

Substituting again for the continuity equation to get an expression involving the two areas, (i.e. $u_2 = u_1 A_1 / A_2$) gives

$$h_L = \left(1 - \frac{A_1}{A_2}\right)^2 \frac{u_1^2}{2g}$$

Equation 20

Comparing this with Equation 14 gives k_L

$$k_L = \left(1 - \frac{A_1}{A_2}\right)^2$$

Equation 21

When a pipe expands in to a large tank $A_1 << A_2$ i.e. $A_1/A_2 = 0$ so $k_L = 1$. That is, the head loss is equal to the velocity head just before the expansion into the tank.

1.5.2 Losses at Sudden Contraction

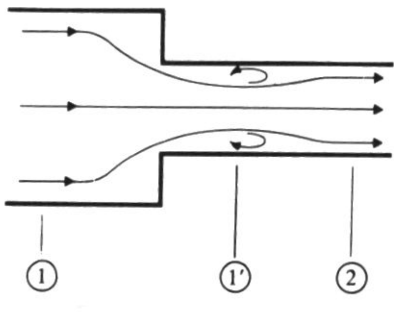

Figure 7: Sudden Contraction

In a sudden contraction, flow contracts from point 1 to point 1', forming a vena contraction. From experiment it has been shown that this contraction is about 40% (i.e. $A_{1'} = 0.6\ A_2$). It is possible to assume that energy losses from 1 to 1' are negligible (no separation occurs in contracting flow) but that major losses occur between 1' and 2 as the flow expands again. In this case Equation 16 can be used from point 1' to 2 to give: (using, by continuity $u_1 = A_2u_2/A_1 = A_2u_2/0.6A_2 = u_2/0.6$)

$$h_L = \left(1 - \frac{0.6A_2}{A_2}\right)^2 \frac{(u_2/0.6)^2}{2g}$$

$$h_L = 0.44\frac{u_2^2}{2g}$$

Equation 22

i.e. At a sudden contraction $k_L = 0.44$.

1.5.3 Other Local Losses

Large losses in energy in energy usually occur only where flow expands. The mechanism at work in these situations is that as velocity decreases (by continuity) so pressure must increase (by Bernoulli).

When the pressure <u>increases</u> in the direction of fluid outside the boundary layer has enough momentum to overcome this pressure that is trying to push it backwards. The fluid within the boundary layer has so little momentum that it will very quickly be <u>brought to rest</u>, and possibly <u>reversed</u> in direction. If this reversal occurs it lifts the boundary layer away from the surface as shown in Figure 8. This phenomenon is known as **boundary layer separation.**

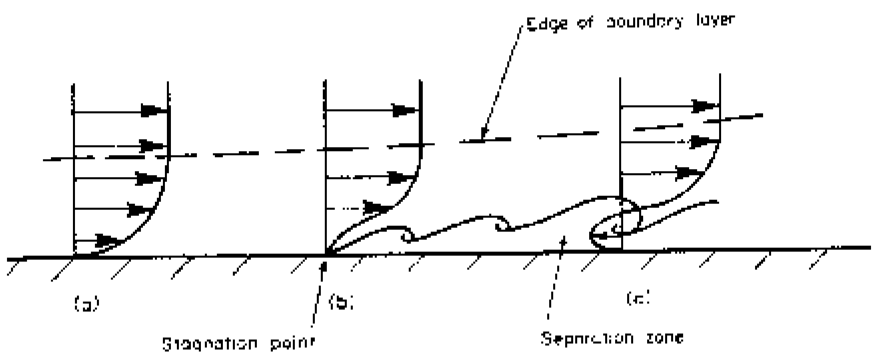

Figure 8: Boundary layer separation

At the edge of the separated boundary layer, where the velocities change direction, a line of vortices occur (known as a vortex sheet). This happens because fluid to either side is moving in the opposite direction. This boundary layer separation and increase in the turbulence because of the vortices results in very large energy losses in the flow. These separating / divergent flows are inherently unstable and far more energy is lost than in parallel or convergent flow.

Some common situation where significant head losses occur in pipe are shown in figure 9

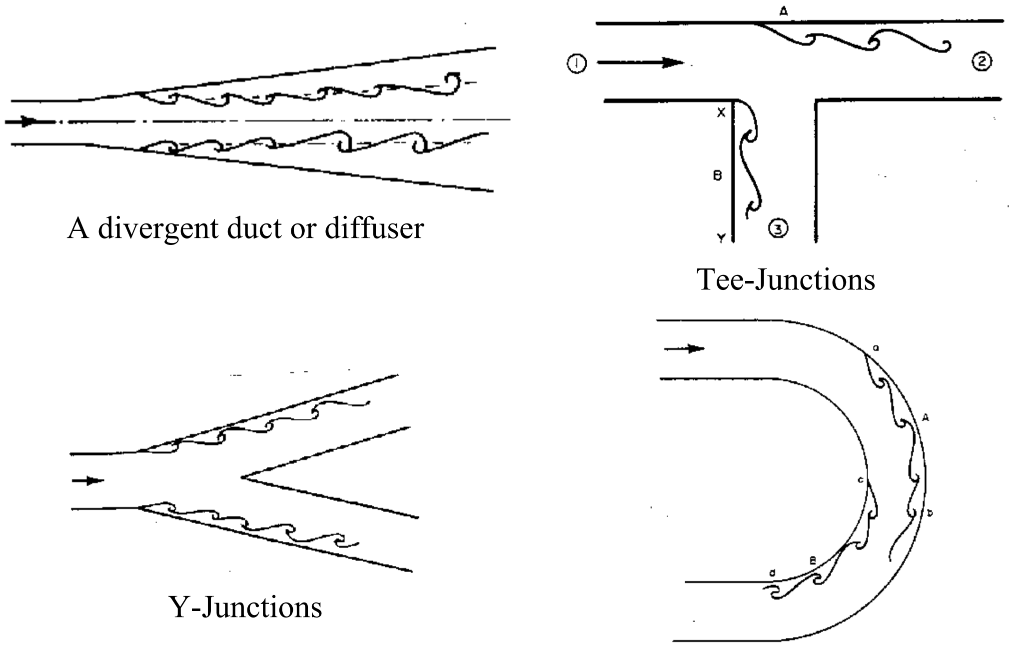

Figure 9: Local losses in pipe flow

The values of k_L for these common situations are shown in Table 2. It gives both the theoretical value and that used in practice.

	k_L value	
	Theory	Practice
Bellmouth entry	0.05	0.10
Sharp entry	0.44	0.5
Sharp exit	0.2	0.5
90° bend	0.4	0.4
90° tees		
In-line flow	0.35	0.4
Branch to line	1.2	1.5
Gate value (open)	0.12	0.25

Table 2: k_L values

1.6 Pipeline Analysis

To analyses the flow in a pipe line we will use Bernoulli's equation. The Bernoulli equation was introduced in the Level 1 module, and as a reminder it is presented again here.

Bernoulli's equation is a statement of conservation of energy along a streamline, by this principle the total *energy* in the system does not change, Thus the total *head* does not change. So the Bernoulli equation can be written

$$\frac{p}{\rho g} + \frac{u^2}{2g} + z = H = \text{constant}$$

or

$$\begin{array}{ccccccc} \text{Pressure} & & \text{Kinetic} & & \text{Potential} & & \text{Total} \\ \text{energy per} & + & \text{energy per} & + & \text{energy per} & = & \text{energy per} \\ \text{unit weight} & & \text{unit weight} & & \text{unit weight} & & \text{unit weight} \end{array}$$

As all of these elements of the equation have units of length, they are often referred to as the following:

pressure head = $\dfrac{p}{\rho g}$

velocity head = $\dfrac{u^2}{2g}$

potential head = z

total head = H

Bernoulli's equation has some restrictions in its applicability, they are:

- Flow is steady;
- Density is constant (i.e. fluid is incompressible);
- Friction losses are negligible.
- The equation relates the states at two points along a single streamline.

1.7 Pressure Head, Velocity Head, Potential Head and Total Head in a Pipeline.

By looking at the example of the reservoir with which feeds a pipe we will see how these different *heads* relate to each other.

Consider the reservoir below feeding a pipe that changes diameter and rises (in reality it may have to pass over a hill) before falling to its final level.

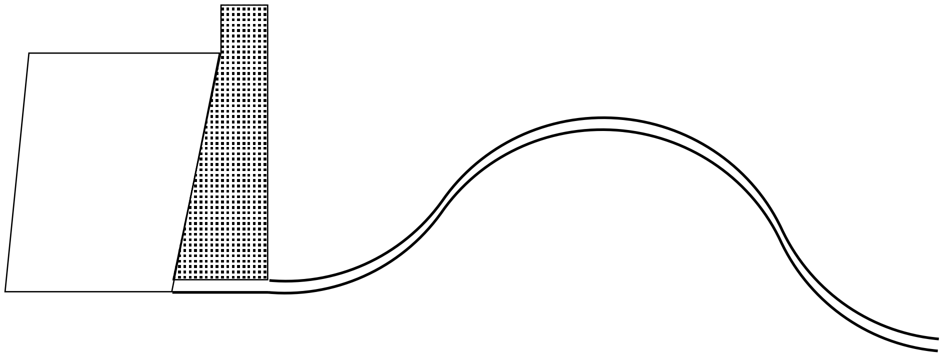

Figure 10: Reservoir feeding a pipe

To analyses the flow in the pipe we apply the Bernoulli equation along a streamline from point 1 on the surface of the reservoir to point 2 at the outlet nozzle of the pipe. And we know that the *total energy per unit weight* or the *total head* does not change - it is **constant** - along a streamline. But what is this value of this constant? We have the Bernoulli equation

$$\frac{p_1}{\rho g} + \frac{u_1^2}{2g} + z_1 = H = \frac{p_2}{\rho g} + \frac{u_2^2}{2g} + z_2$$

We can calculate the total head, *H*, at the reservoir, $p_1 = 0$ as this is atmospheric and atmospheric gauge pressure is zero, the surface is moving very slowly compared to that in the pipe so $u_1 = 0$, so all we are left with is $total\,head = H = z_1$ the elevation of the reservoir.

A useful method of analysing the flow is to show the pressures graphically on the same diagram as the pipe and reservoir. In the figure above the *total head* line is shown. If we attached piezometers at points along the pipe, what would be their levels when the pipe nozzle was closed? (Piezometers, as you will remember, are simply open ended vertical tubes filled with the same liquid whose pressure they are measuring).

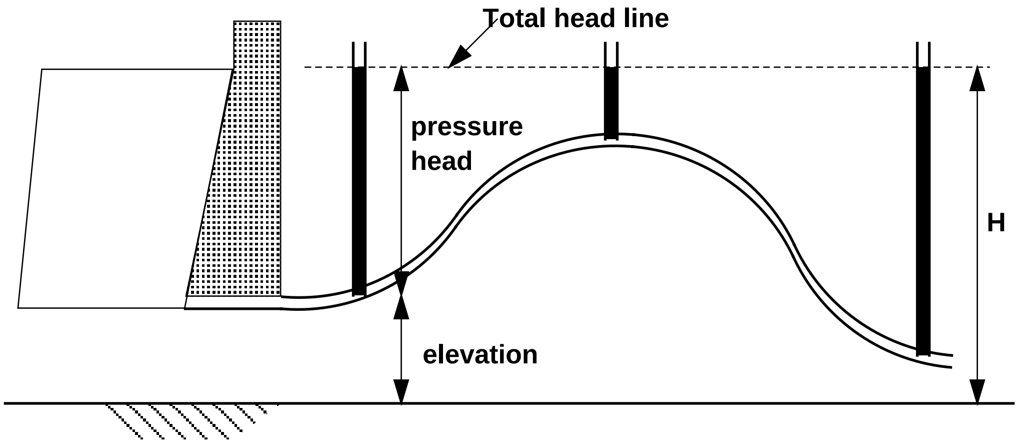

Figure 11: Piezometer levels with zero velocity

As you can see in the above figure, with zero velocity all of the levels in the piezometers are equal and the same as the total head line. At each point on the line, when $u = 0$

$$\frac{p}{\rho g}+z=H$$

The level in the piezometer is the *pressure head* and its value is given by $\frac{p}{\rho g}$.

What would happen to the levels in the piezometers (pressure heads) if the if water was flowing with velocity = u? We know from earlier examples that as velocity increases so pressure falls …

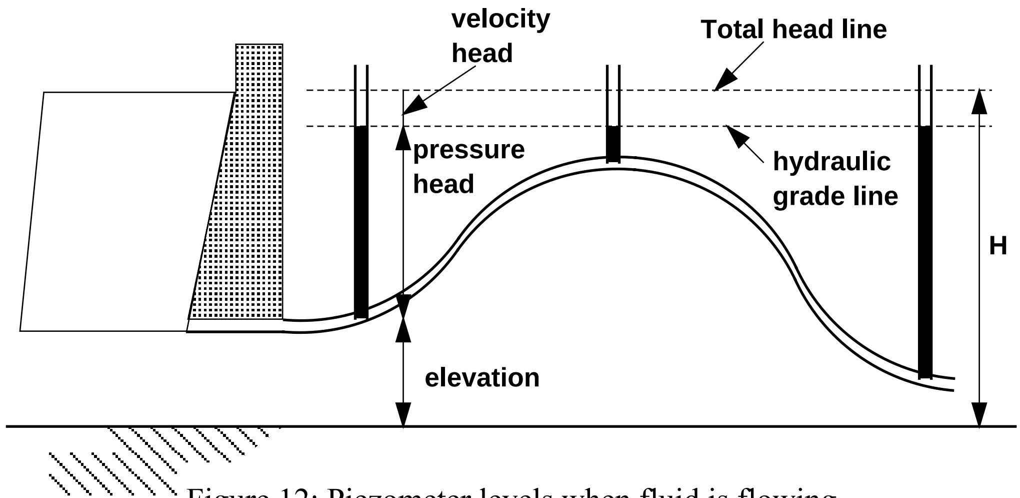

Figure 12: Piezometer levels when fluid is flowing

$$\frac{p}{\rho g}+\frac{u^2}{2g}+z=H$$

We see in this figure that the levels have reduced by an amount equal to the velocity head, $\frac{u^2}{2g}$. Now as the pipe is of constant diameter we know that the velocity is constant along the pipe so the velocity head is constant and represented graphically by the horizontal line shown. (this line is known as the *hydraulic grade line*).

What would happen if the pipe were not of constant diameter? Look at the figure below where the pipe from the example above is replaced by a pipe of three sections with the middle section of larger diameter

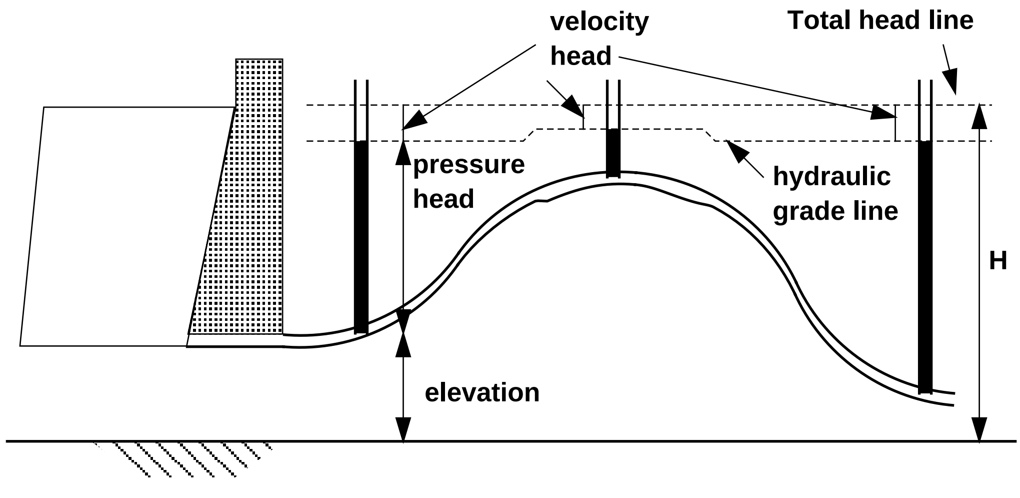

Figure 13: Piezometer levels and velocity heads with fluid flowing in varying diameter pipes

The velocity head at each point is now different. This is because the velocity is different at each point. By considering continuity we know that the velocity is different because the diameter of the pipe is different. Which pipe has the greatest diameter?

Pipe 2, because the velocity, and hence the velocity head, is the smallest.

This graphical representation has the advantage that we can see at a glance the pressures in the system. For example, where along the whole line is the lowest pressure head? It is where the hydraulic grade line is nearest to the pipe elevation i.e. at the highest point of the pipe.

1.8 Flow in pipes with losses due to friction.

In a real pipe line there are energy losses due to friction - these must be taken into account as they can be very significant. How would the pressure and hydraulic grade lines change with friction? Going back to the constant diameter pipe, we would have a pressure situation like this shown below

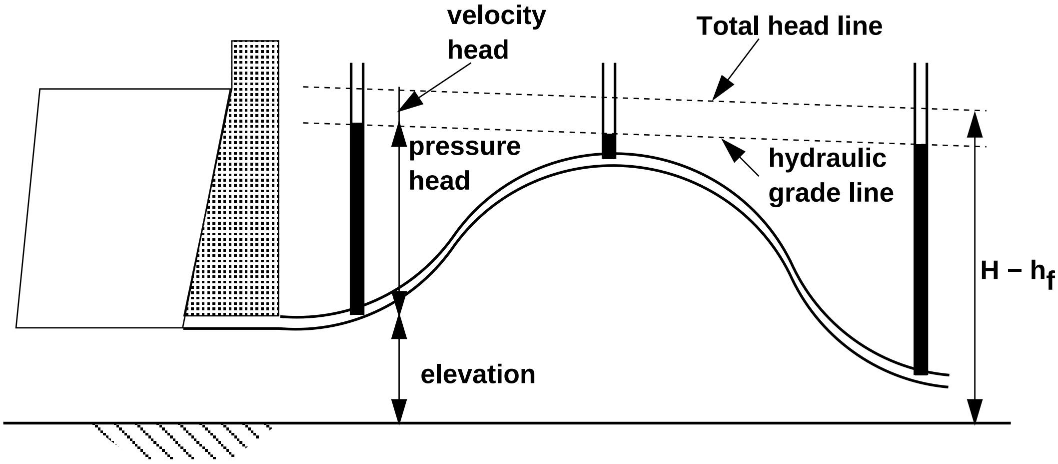

Figure 14: Hydraulic Grade line and Total head lines for a constant diameter pipe with friction

How can the total head be changing? We have said that the total head - or total energy per unit weight - is constant. We are considering energy conservation, so if we allow for an amount of energy to be lost due to friction the total head will change. Equation 19 is the Bernoulli equation as applied to a pipe line with the energy loss due to friction written as a *head* and given the symbol h_f (the *head loss due to friction*) and the local energy losses written as a head, h_L (the local head loss).

$$\frac{p_1}{\rho g}+\frac{u_1^2}{2g}+z_1=\frac{p_2}{\rho g}+\frac{u_2^2}{2g}+z_2+h_f+h_L$$

Equation 23

1.9 Reservoir and Pipe Example

Consider the example of a reservoir feeding a pipe, as shown in figure 15.

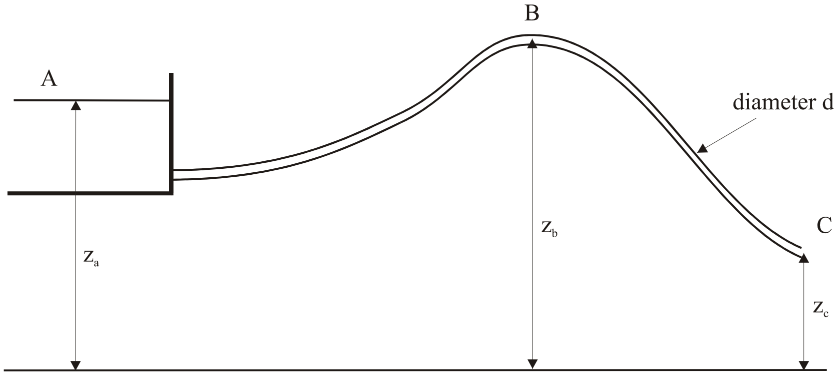

Figure 15: Reservoir feeding a pipe

The pipe diameter is 100mm and has length 15m and feeds directly into the atmosphere at point C 4m below the surface of the reservoir (i.e. $z_a - z_c = 4.0$m). The highest point on the pipe is a B which is 1.5m above the surface of the reservoir (i.e. $z_b - z_a = 1.5$m) and 5 m along the pipe measured from the reservoir. Assume the entrance and exit to the pipe to be sharp and the value of friction factor *f* to be 0.08. Calculate a) velocity of water leaving the pipe at point C, b) pressure in the pipe at point B.

a)
We use the Bernoulli equation with appropriate losses from point A to C
and for entry loss $k_L = 0.5$ and exit loss $k_L = 1.0$.

For the local losses from Table 2 for a sharp entry $k_L = 0.5$ and for the sharp exit as it opens in to the atmosphere with no contraction there are no losses, so

$$h_L = 0.5\frac{u^2}{2g}$$

Friction losses are given by the Darcy equation

$$h_f = \frac{4fLu^2}{2gd}$$

Pressure at A and C are both atmospheric, u_A is very small so can be set to zero, giving

$$z_A = \frac{u^2}{2g} + z_C + \frac{4fLu^2}{2gd} + 0.5\frac{u^2}{2g}$$

$$z_A - z_C = \frac{u^2}{2g}\left(1 + 0.5 + \frac{4fL}{d}\right)$$

Substitute in the numbers from the question

$$4 = \frac{u^2}{2 \times 9.81}\left(1.5 + \frac{4 \times 0.08 \times 15}{0.1}\right)$$

$$u = 1.26 m/s$$

b)
To find the pressure at B apply Bernoulli from point A to B using the velocity calculated above. The length of the pipe is $L_1 = 5$m:

$$z_A = \frac{p_B}{\rho g} + \frac{u^2}{2g} + z_B + \frac{4fL_1u^2}{2gd} + 0.5\frac{u^2}{2g}$$

$$z_A - z_B = \frac{p_B}{\rho g} + \frac{u^2}{2g}\left(1 + 0.5 + \frac{4fL_1}{d}\right)$$

$$-1.5 = \frac{p_B}{1000 \times 9.81} + \frac{1.26^2}{2 \times 9.81}\left(1.5 + \frac{4 \times 0.08 \times 5.0}{0.1}\right)$$

$$p_B = -28.58 \times 10^3 \; N/m^2$$

That is 28.58 kN/m^2 **below** atmospheric.

1.10 Pipes in series

When pipes of different diameters are connected end to end to form a pipe line, they are said to be in series. The total loss of energy (or head) will be the sum of the losses in each pipe plus local losses at connections.

1.10.1 Pipes in Series Example

Consider the two reservoirs shown in figure 16, connected by a single pipe that changes diameter over its length. The surfaces of the two reservoirs have a difference in level of 9m. The pipe has a diameter of 200mm for the first 15m (from A to C) then a diameter of 250mm for the remaining 45m (from C to B).

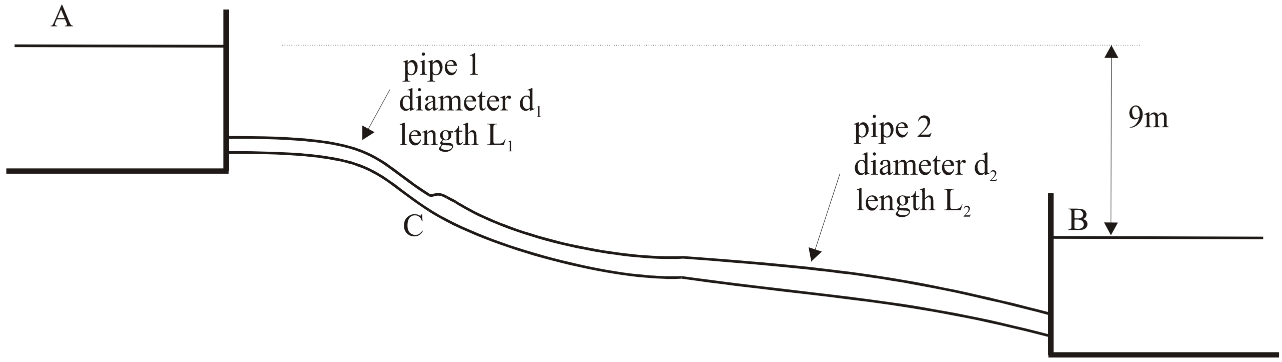

Figure 16:

For the entrance use $k_L = 0.5$ and the exit $k_L = 1.0$. The join at C is sudden. For both pipes use $f = 0.01$.

Total head loss for the system H = height difference of reservoirs

h_{f1} = head loss for 200mm diameter section of pipe
h_{f2} = head loss for 250mm diameter section of pipe
$h_{L\ entry}$ = head loss at entry point
$h_{L\ join}$ = head loss at join of the two pipes
$h_{L\ exit}$ = head loss at exit point

So

$$H = h_{f1} + h_{f2} + h_{L\ entry} + h_{L\ join} + h_{L\ exit} = 9\text{m}$$

All losses are, in terms of Q:

$$h_{f1} = \frac{fL_1Q^2}{3d_1^5}$$

$$h_{f2} = \frac{fL_2Q^2}{3d_2^5}$$

$$h_{Lentry} = 0.5\frac{u_1^2}{2g} = 0.5\frac{1}{2g}\left(\frac{4Q}{\pi d_1^2}\right)^2 = 0.5\times 0.0826\frac{Q^2}{d_1^4} = 0.0413\frac{Q^2}{d_1^4}$$

$$h_{Lexit} = 1.0\frac{u_2^2}{2g} = 1.0\times 0.0826\frac{Q^2}{d_2^4} = 0.0826\frac{Q^2}{d_2^4}$$

$$h_{Ljoin} = \frac{(u_1 - u_2)^2}{2g} = \left(\frac{4Q}{\pi}\right)^2\frac{\left(\frac{1}{d_1^2} - \frac{1}{d_2^2}\right)^2}{2g} = 0.0826Q^2\left(\frac{1}{d_1^2} - \frac{1}{d_2^2}\right)^2$$

Substitute these into

$$h_{f1} + h_{f2} + h_{L\ entry} + h_{L\ join} + h_{L\ exit} = 9$$

and solve for Q, to give $Q = 0.158\ \text{m}^3/\text{s}$

1.11 Pipes in parallel

When two or more pipes in parallel connect two reservoirs, as shown in Figure 17, for example, then the fluid may flow down any of the available pipes at possible different rates. But the head difference over each pipe will always be the same. The total volume flow rate will be the sum of the flow in each pipe.

The analysis can be carried out by simply treating each pipe individually and summing flow rates at the end.

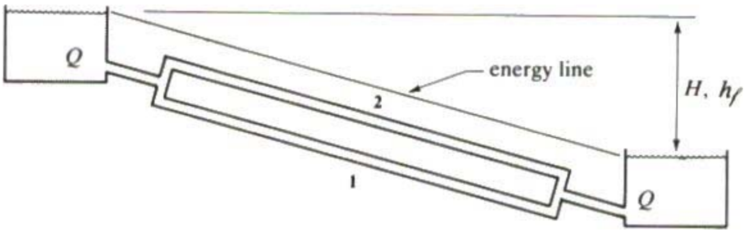

Figure 17: Pipes in Parallel

1.11.1 Pipes in Parallel Example

Two pipes connect two reservoirs (A and B) which have a height difference of 10m. Pipe 1 has diameter 50mm and length 100m. Pipe 2 has diameter 100mm and length 100m. Both have entry loss $k_L = 0.5$ and exit loss k_L=1.0 and Darcy *f* of 0.008.
Calculate:

a) rate of flow for each pipe
b) the diameter D of a pipe 100m long that could replace the two pipes and provide the same flow.

a)
Apply Bernoulli to each pipe separately. For pipe 1:

$$\frac{p_A}{\rho g}+\frac{u_A^2}{2g}+z_A=\frac{p_B}{\rho g}+\frac{u_B^2}{2g}+z_B+0.5\frac{u_1^2}{2g}+\frac{4flu_1^2}{2gd_1}+1.0\frac{u_1^2}{2g}$$

p_A and p_B are atmospheric, and as the reservoir surface move s slowly u_A and u_B are negligible, so

$$z_A-z_B=\left(0.5+\frac{4fl}{d_1}+1.0\right)\frac{u_1^2}{2g}$$

$$10=\left(1.0+\frac{4\times 0.008\times 100}{0.05}\right)\frac{u_1^2}{2\times 9.81}$$

$$u_1=1.731 m/s$$

And flow rate is given by

$$Q_1=u_1\frac{\pi d_1^2}{4}=0.0034 m^3/s$$

For pipe 2:

$$\frac{p_A}{\rho g}+\frac{u_A^2}{2g}+z_A=\frac{p_B}{\rho g}+\frac{u_B^2}{2g}+z_B+0.5\frac{u_2^2}{2g}+\frac{4flu_2^2}{2gd_2}+1.0\frac{u_2^2}{2g}$$

Again p_A and p_B are atmospheric, and as the reservoir surface move s slowly u_A and u_B are negligible, so

$$z_A - z_B = \left(0.5 + \frac{4fl}{d_2} + 1.0\right)\frac{u_2^2}{2g}$$

$$10 = \left(1.0 + \frac{4 \times 0.008 \times 100}{0.1}\right)\frac{u_2^2}{2 \times 9.81}$$

$$u_2 = 2.42\, m/s$$

And flow rate is given by

$$Q_2 = u_2 \frac{\pi d_2^2}{4} = 0.0190\, m^3/s$$

b) Replacing the pipe, we need $Q = Q_1 + Q_2 = 0.0034 + 0.0190 = 0.0224\ m^3/s$

For this pipe, diameter D, velocity u , and making the same assumptions about entry/exit losses, we have:

$$\frac{p_A}{\rho g} + \frac{u_A^2}{2g} + z_A = \frac{p_B}{\rho g} + \frac{u_B^2}{2g} + z_B + 0.5\frac{u^2}{2g} + \frac{4flu^2}{2gD} + 1.0\frac{u^2}{2g}$$

$$z_A - z_B = \left(0.5 + \frac{4fl}{D} + 1.0\right)\frac{u^2}{2g}$$

$$10 = \left(1.0 + \frac{4 \times 0.008 \times 100}{D}\right)\frac{u^2}{2 \times 9.81}$$

$$196.2 = \left(1.0 + \frac{3.2}{D}\right)u^2$$

The velocity can be obtained from Q i.e.

$$Q = Au = \frac{\pi D^2}{4}u$$

$$u = \frac{4Q}{\pi D^2} = \frac{0.02852}{D^2}$$

So

$$196.2 = \left(1.0 + \frac{3.2}{D}\right)\left(\frac{0.02852}{D^2}\right)^2$$

$$0 = 241212 D^5 - 1.5D - 3.2$$

which must be solved iteratively

An approximate answer can be obtained by dropping the second term:

$$0 = 241212 D^5 - 3.2$$

$$D = \sqrt[5]{3.2 / 241212}$$

$$D = 0.1058 m$$

Writing the function

$$f(D) = 241212 D^5 - 1.5D - 3.2$$

$$f(0.1058) = -0.161$$

So increase D slightly, try 0.107m

$$f(0.107) = 0.022$$

i.e. the solution is between 0.107m and 0.1058m but 0.107 if sufficiently accurate.

1.12 Branched pipes

If pipes connect three reservoirs, as shown in Figure 17, then the problem becomes more complex. One of the problems is that it is sometimes difficult to decide which direction fluid will flow. In practice solutions are now done by computer techniques that can determine flow direction, however it is useful to examine the techniques necessary to solve this problem.

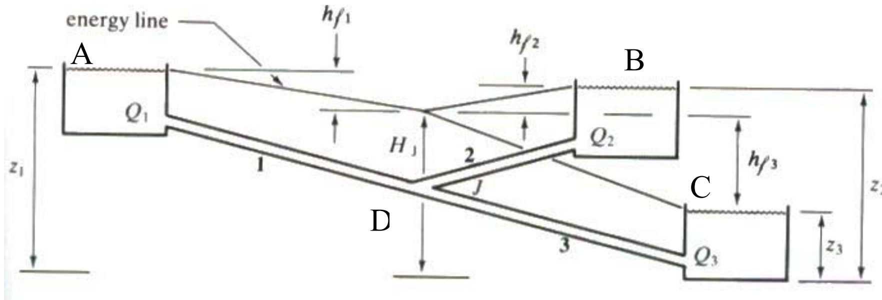

Figure 17: The three reservoir problem

1.12.1 Example of Branched Pipe – The Three Reservoir Problem

Water flows from reservoir A through pipe 1, diameter d_1 = 200mm, length L_1=120m, to junction D from which the two pipes leave, pipe 2, diameter d_2=75mm, length L_2=60m goes to reservoir B, and pipe 3, diameter d_3=60mm, length L_3=40m goes to reservoir C. Reservoir B is 16m below reservoir A, and reservoir C is 24m below reservoir A. All pipes have $f = 0.01$. (Ignore and entry and exit losses.)

In this case the flow will be from reservoir A to junction D then from D to reservoirs B and C. There are three unknowns u_1, u_2 and u_3 the three equation we need to solve are obtained from A to B then A to C and from continuity at the junction D.

Flow from A to B

$$\frac{p_A}{\rho g}+\frac{u_A^2}{2g}+z_A=\frac{p_B}{\rho g}+\frac{u_B^2}{2g}+z_B+\frac{4fL_2u_1^2}{2gd_1}+\frac{4fL_2u_2^2}{2gd_2}$$

Putting $p_A = p_B$ and taking u_A and u_B as negligible, gives

$$z_A - z_B = \frac{4fL_2u_1^2}{2gd_1}+\frac{4fL_2u_2^2}{2gd_2}$$

Put in the numbers from the question

$$16 = \frac{4\times 0.01\times 120u_1^2}{2g0.12}+\frac{4\times 0.01\times 60u_2^2}{2g0.075}$$

$$16 = 2.0387u_1^2 + 1.6310u_2^2$$

(equation i)

Flow from A to C

$$\frac{p_A}{\rho g}+\frac{u_A^2}{2g}+z_A=\frac{p_C}{\rho g}+\frac{u_C^2}{2g}+z_C+\frac{4fL_2u_1^2}{2gd_1}+\frac{4fL_3u_3^2}{2gd_3}$$

Putting $p_A = p_c$ and taking u_A and u_c as negligible, gives

$$z_A - z_C = \frac{4fL_2u_1^2}{2gd_1}+\frac{4fL_3u_3^2}{2gd_3}$$

Put in the numbers from the question

$$24 = \frac{4 \times 0.01 \times 120 u_1^2}{2g0.12} + \frac{4 \times 0.01 \times 40 u_3^2}{2g0.060}$$

$$24 = 2.0387 u_1^2 + 1.3592 u_3^2$$

(equation ii)

Fro continuity at the junction

Flow A to D = Flow D to B + Flow D to C

$$Q_1 = Q_2 + Q_3$$

$$\frac{\pi d_1^2}{4} u_1 = \frac{\pi d_2^2}{4} u_2 + \frac{\pi d_3^2}{4} u_3$$

$$u_1 = \left(\frac{d_2}{d_1}\right)^2 u_2 + \left(\frac{d_3}{d_1}\right)^2 u_3$$

with numbers from the question

$$u_1 - 0.3906 u_2 - 0.25 u_3 = 0$$

(equation iii)

the values of u_1, u_2 and u_3 must be found by solving the simultaneous equation i, ii and iii. The technique to do this is to substitute for equations i, and ii in to equation iii, then solve this expression. It is usually done by a trial and error approach.
i.e. from i,

$$u_2 = \sqrt{9.81 - 1.25 u_1^2}$$

from ii,

$$u_3 = \sqrt{17.657 - 1.5 u_1^2}$$

substituted in iii gives

$$u_1 - 0.3906\sqrt{9.81 - 1.25 u_1^2} - 0.25\sqrt{17.657 - 1.5 u_1^2} = 0 = f(u_1)$$

This table shows some trial and error solutions

u	f(u)
1	-1.14769
2	0.289789
1.8	-0.03176
1.85	0.046606
1.83	0.015107
1.82	-0.00057

Giving u_1 = 1.82 m/s, so u_2 = 2.38 m/s, u_3 = 12.69 m/s
Flow rates are

$$Q_1 = \frac{\pi d_1^2}{4} u_1 = 0.0206\, m^3/s$$

$$Q_2 = \frac{\pi d_2^2}{4} u_2 = 0.0105\, m^3/s$$

$$Q_3 = \frac{\pi d_3^2}{4} u_3 = 0.0101\, m^3/s$$

Check for continuity at the junction

$$Q_1 = Q_2 + Q_3$$

$$0.0206 = 0.0105 + 0.0101$$